Questions for you:

1. Suppose f is holomorphic, and
$$f\left(\frac{i}{n}\right) = \frac{1}{\frac{i^3}{n^2}+1} \quad \text{for } n \in \mathbb{N}.$$
Find all possible values of $f(5)$.

# of values	
0	1
1	4
2	1

2. Let f be holomorphic on $\overline{B(0,1)}$ and not $\equiv 0$. Is it possible that f has an infinite # of zeros in $\overline{B(0,1)}$?

Y	N
2	3
1	both

3. Let f be holomorphic on $B(0,1)$ and not $\equiv 0$. Is it possible that f has an infinite # of zeros in $B(0,1)$?

Y	N
3	2
both	

4. Let f be holomorphic on $\overline{B(0,1)} \setminus \{0\}$ and not $\equiv 0$. Is it possible that f has an infinite # of zeros in $\overline{B(0,1)} \setminus \{0\}$?

Y	N
~~4~~	5
2	~~both~~

---

1. $$f\left(\frac{i}{n}\right) = \frac{1}{\left(\frac{i}{n}\right)^2 i + 1}$$

$\therefore\ g(z) = \dfrac{1}{z^2 i + 1}$ is one fcn for which that equation is true, $\forall n \in \mathbb{N}$.

$\lim_{n\to\infty} \frac{i}{n} = 0 \in \text{domain}(g)$.

$\therefore$ By the uniqueness thm,
g is the only holomorphic fcn that satisfies that equation. $\therefore f = g$.

$$\therefore f(5) = \frac{1}{5^2 i + 1} = \frac{1}{25i+1}.$$

---

(2) No, because if there is an ∞ # of zeros, $\exists$ a sequence of them that converges to a point of $\overline{B(0,1)}$ (because it's compact), and that point is in the domain of f.
By the identity thm $f \equiv 0$ on $\overline{B(0,1)}$
(in fact its connected component of its domain.)

---

(3) Let $f(z) = \sin\left(\pi \frac{1}{1-z}\right)$

Let $z_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$

$$\sin\left(\pi \left(\frac{1}{1-(1-\frac{1}{n})}\right)\right) = \sin\left(\pi \frac{1}{\frac{1}{n}}\right)$$

$= \sin(n\pi) = 0$.

f is holom on $\mathbb{C} \setminus \{1\}$

$\cup\!/$

$B(0,1)$.

This provides an example.

---

④ Let $f(z) = \sin\left(\frac{\pi}{z}\right)$.

Let $z_n = \frac{1}{n} \Rightarrow$

$$f(z_n) = \sin\left(\frac{\pi}{\left(\frac{1}{n}\right)}\right) = \sin(n\pi)$$

$$= 0.$$

$$\forall n \in \mathbb{N}.$$

$\therefore \operatorname{Dom}(f) = \mathbb{C} \setminus \{0\}$

$$\overline{B(0,1)} \setminus \{0\} \subseteq \mathbb{C} \setminus \{0\}.$$

This provides an example.

Example of a Taylor series that does not converge at a certain point.

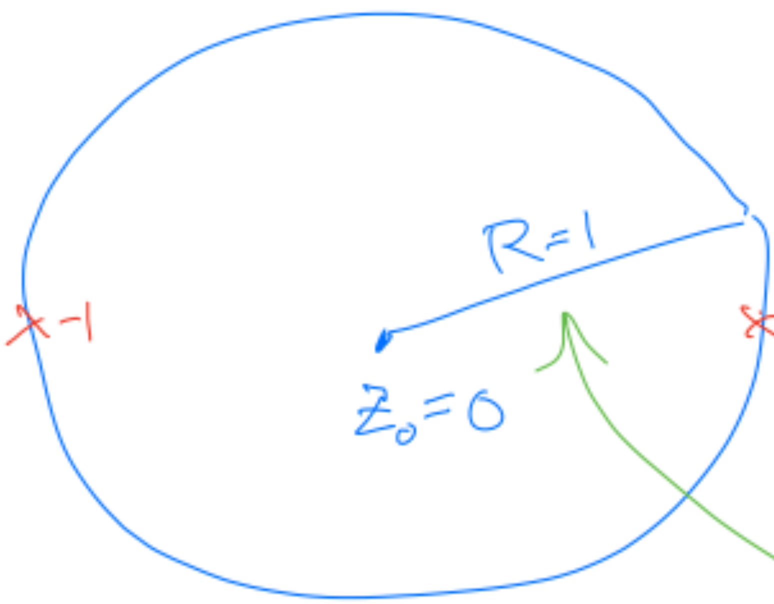
$$f(z) = \frac{z}{1-z^2} \longrightarrow \text{Taylor series centered at } 0$$

$$= z \cdot \frac{1}{1-z^2} = z \cdot \sum_{k=0}^{\infty} (z^2)^k$$

converges $\Leftrightarrow |z^2| < 1 \Leftrightarrow |z| < 1.$

$$f(z) = \sum_{k=0}^{\infty} z^{2k+1}$$

Converges on $B(0,1)$.



f blows up at $z=1, z=-1$.

The radius of convergence of the T.S. of f is the distance from the center z_0 to the nearest singularity.

or the function can not be extended to a holomorphic function $B(z_0, R+\varepsilon)$ for any $\varepsilon > 0$.

($R =$ maximum # s.t. f can be defined as holomorphic on $B(z_0, R)$.)

14.10 :

14.10 True or False (Provide justification.)

(a) If $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ is analytic on a region containing $\{z : |z - z_0| \leq R\}$, then there is a positive integer M such that $|a_n| \leq \frac{M}{R^n}$ for all $n \geq 0$.

(b) If $\sum_{n \geq 0} a_n (z - z_0)^n$ converges on $\{z : |z - z_0| < R\}$ and diverges on at least one point of $\{z : |z - z_0| = R\}$, then there is a positive integer M such that $|a_n| \geq \frac{M}{R^n}$ for all $n \geq 0$.

Perhaps some $a_n = 0$?

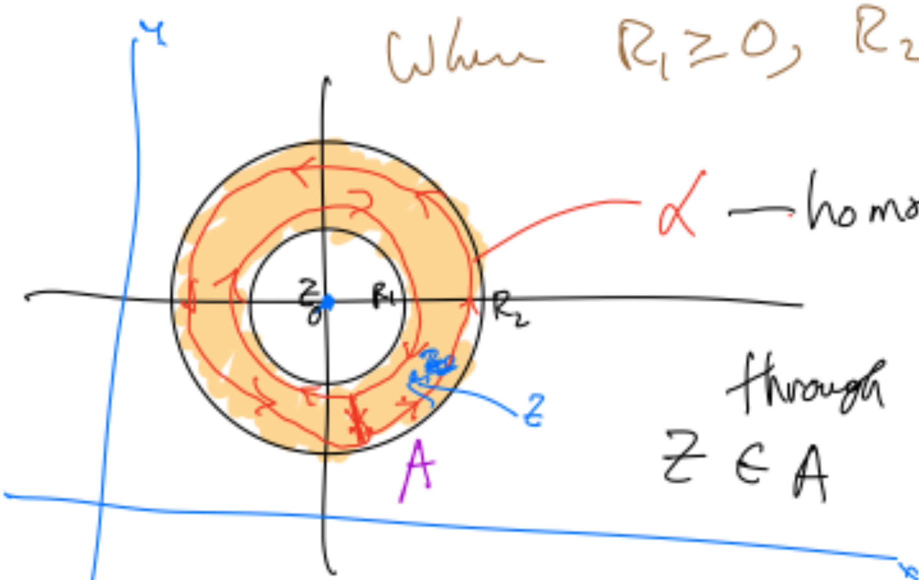
(c) If f is a holomorphic function on an open set U in $\mathbf{C}$, then for every $z_0 \in U$, there is a positive number ρ so that the Taylor series of f centered at z_0 converges uniformly on the set $\{z : |z - z_0| < \rho\}$.

Important:

If R is the radius of convergence of $\sum_{k \geq 0} c_k z^k$, then if $0 < r < R$, then $\sum_{k \geq 0} c_k z^k$ converges absolutely and uniformly on $\overline{B(0, r)}$.

New kind of Series:

Suppose f is a holomorphic function that is defined on an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ when $R_1 \geq 0$, $R_2 > R_1 \geq 0$.

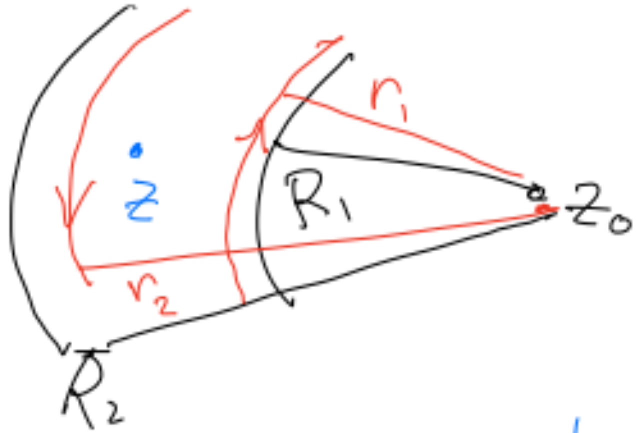


α — homotopic to $\partial B(z, \varepsilon) = \tilde{\alpha}$ through domain A.

$z \in A$

$$f(z) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(w)}{w - z} \, dw$$

Now, $\displaystyle \int_{\alpha} = \int_{C_{r_2}(z_0)} + -\int_{C_{r_1}(z_0)}$



When $R_1 < r_1 < |z - z_0| < r_2 < R_2$

$$\therefore \quad f(z) = \frac{1}{2\pi i}\int_{\alpha} \frac{f(\omega)}{\omega - z}\,d\omega =$$

$$\frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(\omega)}{\omega - z}\,d\omega - \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(\omega)}{\omega - z}\,d\omega$$

by deformation Thm.